XX Intersection theory

§1 Arakelov's theory

$X/\mathbb{Q}$ sm proj curve, $\mathcal{X}/\operatorname{Spec}\mathbb{Z}$ flat proj and regular

Recall: $\widehat{CH}^1 = \{(D,g) \mid D \in \operatorname{Div}(\mathcal{X}),\ g \in D^{1,1}(X) \text{ GF for } D_{\mathbb{C}}\} / \text{principal divisors}$

Thm. $\exists$ intersection pairing $\widehat{CH}^1_{\mathbb{Q}} \otimes \widehat{CH}^1_{\mathbb{Q}} \xrightarrow{\ \cdot\ } \widehat{CH}^2_{\mathbb{Q}} \xrightarrow{\ \deg\ } \mathbb{R}$, symmetric and extends the usual pairing on $\widehat{CH}(\mathcal{X})_{\mathbb{Q}}$.

Construction. (D_i, g_i), $i = 1,2$

- Replacing by rat equivalent cycle, wma $|D_{1,\mathbb{Q}}| \cap |D_{2,\mathbb{Q}}| = \emptyset$
- Vertical divisor := elt of $\ker(\operatorname{Div}(\mathcal{X}) \to \operatorname{Div}(X))$

  $D_i = D_{i,\text{vert}} + D_{i,\text{hor}}$

By linearity and symmetry, it suffices to construct

1) vert $\cdot$ hor; 2) vert $\cdot$ vert; 3) hor $\cdot$ hor.

1) $D_{1,\text{vert}} = \sum_i n_i V_i \qquad D_{2,\text{hor}} = \sum_j m_j H_j$

$$(D_{1,\text{vert}}, \ldots) \cdot (D_{2,\text{hor}}, g_2) = \sum_{i,j,k} n_i m_j \log \left| \operatorname{Tor}_k^{\mathcal{O}_\mathcal{X}} (\mathcal{O}_{V_i}, \mathcal{O}_{H_j}) \right|^{(-1)^k}$$

$$= \sum_{p<\infty} \log p \cdot \sum_{i,j,k} n_i m_j \operatorname{len}_{\mathbb{Z}} \left(\operatorname{Tor}_k^{\mathcal{O}_\mathcal{X}} (\mathcal{O}_{V_i}, \mathcal{O}_{H_j}) \right)^{(-1)^k}$$

$V_i \longrightarrow \operatorname{Spec}\mathbb{F}_p$

2) $D_{1,\text{vert}} = \sum_i n_i V_i \qquad D_{2,\text{vert}} = \sum_j m_j W_j \qquad V_i W_j = \begin{cases} \text{as above} & \text{if } V_i \neq W_j \\ @ & \text{if } V_i = W_j \end{cases}$

@: Want to def $V \cdot V$ for $V \in \mathcal{X} \otimes_{\mathbb{Z}} \mathbb{F}_p$ irred component.
Write $\mathcal{X} \otimes_{\mathbb{Z}} \mathbb{F}_p = \sum_{i=0}^{r} e_i W_i$ with $e_i \in \mathbb{N}$, $W_0 = V$.
Then by rat equiv, $(V, 0) \sim e_0^{-1} \left(-\sum_{i=1}^{r} e_i W_i, + \log|p|^2 \right)$

$$\Rightarrow V^2 = e_0^{-1} \sum_{i=1}^{r} (V, e_i W_i)$$

3) D_1, D_2 horizontal, $|D_{1,\mathbb{Q}}| \cap |D_{2,\mathbb{Q}}| = \emptyset$

$$(D_1, g_1) \cdot (D_2, g_2) := \left(g_2 \wedge \delta_{D_{1,\mathbb{Q}}} + \omega_{(D_2,g_2)} \wedge g_1 \right)(1) = \delta_{D_{2,\mathbb{Q}}}(g_1) + \int_{X(\mathbb{C})} \omega_{(D_2,g_2)} \cdot g_1$$

Motivation. (Y, g_Y), (Z, g_Z) cycles + GC on a proj cx mf X, Y, Z intersect properly

$*$-product: $g_Y * g_Z := g_Y \hat{*} \delta_Z + \omega_{g_Y, Y} \wedge g_Z$

Formally, this is a GC for $Y \cdot Z$:

$$dd^c(g_Y * g_Z) = dd^c g_Y \wedge \delta_Z + \omega_{g_Y, Y} \wedge dd^c g_Z$$
$$= (\omega_Y - \delta_Y) \wedge \delta_Z + \omega_Y \wedge (\omega_Z - \delta_Z)$$
$$= -\delta_Y \wedge \delta_Z + \omega_Y \wedge \omega_Z$$
$$= -\delta_{Y \cdot Z} + \omega_Y \wedge \omega_Z$$

§2 Thuillier's theory

$X / \mathbb{C}_p$ smooth proj curve

Recall: $\varphi : |X^{an}| \to \mathbb{R}$ pw lin $\Rightarrow dd^c \varphi = \sum_{x\,(2)} \sum_{v \in T_x} d_v \varphi(x) \cdot \delta_x \in TA^{1,1}(X^{an})$

GC for a divisor D: $g : X^{an} \to \mathbb{R} \cup \{\infty\}$ s.t. • $g|_{X^{an} \setminus D}$ is pw lin

• $dd^c g + \delta_D = [\omega_{(D,g)}]$ for some $\omega_{D,g} \in TA^{1,1}$

Def. (D_i, g_i), $i = 1, 2$. The $*$-product: $g_1 * g_2 = g_1 \cdot \delta_{D_2} + \omega_{(D_1, g_1)} \cdot g_2 \in D^{1,1}$ when $|D_1| \cap |D_2| = \emptyset$.

Def. Intersection number: $(g_1 * g_2)(1)$

Prop. (Thuillier 4.1.4) $g_1 * g_2 = g_2 * g_1$ for (D_i, g_i) as above.

Let $\mathfrak{X} / \mathrm{Spf}\, \mathcal{O}_{\mathbb{C}_p}$ be a strongly sst model of X, $|D_1| \cap |D_2| = \emptyset$

$\mathcal{D}_i \in \mathrm{Div}(\mathfrak{X})$ flat closure.

(If $D_1 = \sum_P n_P P$ then $\mathcal{D}_1 = \sum_P n_P \overline{\{P\}}$, $\overline{\{P\}} : \mathrm{Spf}\, \mathcal{O}_{\mathbb{C}_p} \hookrightarrow \mathfrak{X}$ closed immersion

In particular, $\mathrm{Tor}_k^{\mathcal{O}_\mathfrak{X}}(\mathcal{O}_{\overline{\{P_1\}}}, \mathcal{O}_{\overline{\{P_2\}}}) = 0$ for $k > 0$. $P_1 \neq P_2$)

$\mathcal{L}_i := \mathcal{O}_\mathfrak{X}(\mathcal{D}_i)$ formal model for $\mathcal{O}_X(D_i)$

$1 : \mathcal{O}_\mathfrak{X} \to \mathcal{L}_i$ meromorphic section

Prop: $|D_1| \cap |D_2| \neq \emptyset$, $g_i := -\log \|1\|_{\mathcal{L}_i}$ $\Rightarrow (g_1 * g_2)(1) = \langle \mathcal{D}_1, \mathcal{D}_2 \rangle_\mathfrak{X}$ where

$\langle \overline{\{P_1\}}, \overline{\{P_2\}} \rangle := v(1)$ if $\mathcal{O}_{\overline{\{P_1\}}} \otimes_{\mathcal{O}_\mathfrak{X}} \mathcal{O}_{\overline{\{P_2\}}} \simeq \mathcal{O}_{\mathbb{C}_p}/(\ldots)$ and ..

Thm. $\mathcal{X}$ strongly sst flat proj, reg model of X over Spec $\mathbb{Z}$. Then
$(Z, g_\infty) \mapsto \left(Z := Z_{\mathbb{Q}}, \left(-\log \|1\|_{\mathcal{O}_{\mathcal{X}_p}(Z_p)}\right)_{p<\infty}, g_\infty\right)$ defines an injection
$\hat{Z}^1(\mathcal{X}) \hookrightarrow \widehat{TZ}^1(X)$ descending to $\widehat{CH}^1(\mathcal{X}) \to \widehat{TCH}^1(X)$ commuting
with intersection product.

XX | Néron–Tate height & Outlook

11.01.2019

[Kawaguchi, Moriwaki, Yamaki: Intro to Arakelov Theory]

§1 Heights

$\mathcal{X} \to \operatorname{Spec} \mathcal{O}_K$ flat proj, generically smooth,

$K/\mathbb{Q}$ number field

$\bar{\mathcal{L}} = (\mathcal{L}, \|\cdot\|)$ metrised lb on $\mathcal{X}$

$\forall \sigma: K \to \mathbb{C}$, smooth metric on $X_\sigma(\mathbb{C})$ where $X = \mathcal{X} \times_{\operatorname{Spec} \mathcal{O}_K} \operatorname{Spec} K$

Def. Height induced by $\bar{\mathcal{L}}$: $h_{(\mathcal{X}, \bar{\mathcal{L}})}: X(\bar{K}) \to \mathbb{R}$.
Given $x \in X(\bar{K})$ let $K(x)/K$ be the residue field. Better: $x \in X(K(x))$ with $K(x)/K$ finite
By the val. criterion we get $\operatorname{Spec} \mathcal{O}_{K(x)} \xrightarrow{\tilde{x}} \mathcal{X}$.
Then $\tilde{x}^* \bar{\mathcal{L}}$ proj rk 1 $\mathcal{O}_{K(x)}$-module, and $\forall \sigma: K(x) \hookrightarrow \mathbb{C}$: $\|\cdot\|_\sigma$ on
$(\tilde{x}^* \bar{\mathcal{L}}) \otimes_{\mathcal{O}_{K(x)}, \sigma} \mathbb{C}$.

Now let $s \in \tilde{x}^* \bar{\mathcal{L}} \setminus \{0\}$ be arbitrary, and define

$$h_{(\mathcal{X}, \bar{\mathcal{L}})}(x) := \left(\log \left| \tilde{x}^* \bar{\mathcal{L}} / \mathcal{O}_{K(x)} \cdot s \right| - \sum_{\sigma: K(x) \hookrightarrow \mathbb{C}} \log \|s\|_\sigma \right) \cdot [K(x) : K]^{-1}$$

Exc. $\widehat{\operatorname{Pic}}(\mathcal{X}) :=$ ab gp of $(\mathcal{L}, \|\cdot\|)$ up to iso. Then for $x \in X(\bar{K})$ fixed,
$h_{\_}(x): \widehat{\operatorname{Pic}}(\mathcal{X}) \to \mathbb{R}$ is a group homomorphism.

Prop. Let $L := \mathcal{L}|_X$ and $\operatorname{Bs}(L) :=$ base point locus $:= \operatorname{Supp} \operatorname{Coker}\left(\mathcal{O}_X \otimes_K H^0(X, L) \to L\right)$.
Then $\exists C \in \mathbb{R}$ s.t. $h_{(\mathcal{X}, \mathcal{L})}(x) \geq C \quad \forall x \in (X \setminus \operatorname{Bs}(L))(\bar{K})$

Pf: Let $s_1, \dots, s_n$ be $\mathcal{O}_K$-generators of $H^0(\mathcal{X}, \mathcal{L})$.
$a := \max \{ \sup \log \|s_i\|_\sigma \mid \sigma: K \hookrightarrow \mathbb{C} \}$
Then for $x \in (X \setminus \operatorname{Bs}(L))(\bar{K})$ $\exists i$: $s_i(x) \neq 0$. Hence $h_{(\mathcal{X}, \bar{\mathcal{L}})}(x) \geq [K : \mathbb{Q}] \cdot a$

Cor. If $L \simeq \mathcal{O}_X$ then $\exists C$ s.t. $-C \le \left| h_{(\mathcal{X}, \bar{\mathcal{L}})}(x) \right| \le C$

Pf: Apply Prop. to $\bar{\mathcal{L}}$ and $\bar{\mathcal{L}}^{-1}$

Cor. Let $(\mathcal{X}, \bar{\mathcal{L}})$ and $(\mathcal{X}', \bar{\mathcal{L}}')$ be as above, models of the same X, i.e.

$$\mathcal{X} \otimes_{\mathcal{O}_K} K \simeq X \simeq \mathcal{X}' \otimes_{\mathcal{O}_K} K \quad \text{and} \quad \bar{\mathcal{L}}|_X \simeq L \simeq \bar{\mathcal{L}}'|_X .$$

Then $\exists C$: $\left| h_{(\mathcal{X}, \bar{\mathcal{L}})}(x) - h_{(\mathcal{X}', \bar{\mathcal{L}}')}(x) \right| \le C \quad \forall x \in X(\bar{K})$

Pf: $\mathcal{X} \dashrightarrow \mathcal{X}'$ birational map.

$\mathcal{X}'' \subseteq \mathcal{X} \otimes_{\operatorname{Spec} \mathcal{O}_K} \mathcal{X}'$ closure of graph

$$\begin{tikzcd} & \mathcal{X}'' \arrow[dl, "\mu"'] \arrow[dr, "\mu'"] & \\ \mathcal{X} & & \mathcal{X}' \end{tikzcd}$$

Then $h_{(\mathcal{X}'', \mu^* \bar{\mathcal{L}})} = h_{(\mathcal{X}, \bar{\mathcal{L}})}$ and

$h_{(\mathcal{X}'', \mu^* \bar{\mathcal{L}}')} = h_{(\mathcal{X}', \bar{\mathcal{L}}')}$

On $\mathcal{X}''$, the claim follows from prev. Cor and considering $\mu^* \bar{\mathcal{L}} \otimes (\mu'^* \bar{\mathcal{L}}')^{-1}$. □

Upshot: $\operatorname{BMap}(X(\bar{K}), \mathbb{R}) := \{ f : X(\bar{K}) \to \mathbb{R} \mid \exists C : |f| \le C \}$ bounded maps,

Def.
$$\operatorname{Pic}(X) \longrightarrow \operatorname{Map}(X(\bar{K}), \mathbb{R}) \big/ \operatorname{BMap}(X(\bar{K}), \mathbb{R})$$

$$L \longmapsto h_L := \left[h_{(\mathcal{X}, \bar{\mathcal{L}})} \right]$$ for some model $\mathcal{X}$ of X, $\mathcal{L}$ of L, $\|\cdot\|$ metric on $\mathcal{L}$.

Thm. (Northcott) L ample, $d, M \ge 0$. Then $\{ x \in X(\bar{K}) \mid [K(x) : K] \le d,\ h_L(x) \le M \}$ is finite where h_L is some fixed representative under the map above.

Pf (SKETCH): [Faltings' article, Wüstholz "Rational pts" 1983/84]

- L ample $\Rightarrow$ we may replace by $L^{\otimes n}$, so wma L is very ample, i.e. $i^* \mathcal{O}(1)$ for some $i : X \hookrightarrow \mathbb{P}^N$.
- By previous results, up to a bounded function, h_L is the usual ht on $\mathbb{P}^N$, defined as follows: $h([x_0 : \dots : x_N]) := \frac{1}{[F:\mathbb{Q}]} \cdot \sum_{v \in \Sigma_F} \max \{ \log |x_i|_v^{\varepsilon_v} \}$

  where F is some number field s.t. $\forall x_i \in F$. $\varepsilon_v := \begin{cases} 1 & v \text{ non-arch or } F_v \simeq \mathbb{R} \\ 2 & F_v \simeq \mathbb{C} \end{cases}$
- For $[x_0 : \dots : x_N] \in \mathbb{P}^N(K(x))$ with $[K(x) : \mathbb{Q}] < d$ let $[X_0 : \dots : X_n]$ be the coefficient of norm form $\operatorname{Nm}_{K(x)/\mathbb{Q}} \left(\sum_{i=1}^N T_i x_i \right)$
- It remains to check that bounded pts get mapped to bd pts. □

§2 Canonical heights

$K/\mathbb{Q}$ number field

A/K abelian vty

L lb on A, h_L associated height

We want a good representation

Note. $\mathcal{X} \xrightarrow{u} \mathcal{Y} / \operatorname{Spec} \mathcal{O}_K$, $\bar{\mathcal{L}}$ on $\mathcal{Y}$, $x \in X(\bar{K})$. Then $h_{(\mathcal{X}, u^* \bar{\mathcal{L}})}(x) = h_{(\mathcal{Y}, \bar{\mathcal{L}})}(u \circ x)$.

Lemma. On $(A \times A \times A)(K)$,

$$(x,y,z) \mapsto h_L(x+y+z) - h_L(x+y) - h_L(y+z) - h_L(x+z) + h_L(x) + h_L(y) + h_L(z) \sim 0,$$

i.e. it is a bounded map.

Pf: Thm. of Cube and the Note on functoriality.

Thm of the Cube. $\forall L$ on A: on $A \times A \times A$, (m_3 and m_2 are the multiplication maps)

$$m_3^* L \otimes p_{12}^* m_2^* L^{-1} \otimes p_{13}^* m_2^* L^{-1} \otimes p_{23}^* m_2^* L^{-1} \otimes p_1^* L \otimes p_2^* L \otimes p_3^* L \simeq \mathcal{O}_{A \times A \times A}$$

Lemma. (Tate) G ab gp, $f: G \longrightarrow \mathbb{R}$, $f(x,y,z) - \dots + f(z) \sim 0$ (i.e. f is bounded) then $\exists q$ quadratic form, $\exists \ell$ linear form s.t. $f \sim q + \ell$.

Pf: Idea: $\beta(x_1, x_2) := f(x_1, x_2) - f(x_1) - f(x_2) \Rightarrow \beta(x_1, x_2, x_3) \sim \beta(x_1, x_2) + \beta(x_2, x_3)$

$b(x_1, x_2) :\overset{\exists}{=} \lim_{n \to \infty} 4^{-n} \beta(2^n x_1, 2^n x_2)$: $G \times G \longrightarrow \mathbb{R}$ bilinear form on G

Similarly: $\lambda(x) := f(x) - \frac{1}{2} b(x,x) \Rightarrow \lambda(x_1 + x_2) - \lambda(x_1) - \lambda(x_2) \sim \beta(x_1, x_2) - b(x_1, x_2) \sim 0$.

$\Rightarrow \ell(x) :\overset{\exists}{=} \lim_{n \to \infty} 2^{-n} \lambda(2^n x)$: $G \longrightarrow \mathbb{R}$ linear form

Def. A/K ab var, L lb on A. Then the Néron–Tate height is $\hat{h}_L := \underbrace{q_L + \ell_L}_{\text{canonically constructed from } h_L}$.

If L is symmetric, i.e. $[-1]^* L = L$ then $\ell_L = 0$.

$\Rightarrow \hat{h}_L$ is a quadratic form $A(\bar{K}) \longrightarrow \mathbb{R}$.

Néron–Tate height pairing: $A(\bar{K}) \times A(\bar{K}) \longrightarrow \mathbb{R}$

$$(x,y) \longmapsto \hat{h}_L(x+y) - \hat{h}_L(x) - \hat{h}_L(y)$$

Prop. L symm + ample. Then

1) $\forall x$: $\hat{h}_L(x) \geq 0$

2) $\hat{h}_L(x) = 0 \iff x \in A(\bar{K})_{\text{tors}}$.

Pf: 1) $\hat{h}_{L^{\otimes n}} = n\,\hat{h}_L \Rightarrow$ wlog L glob generated

By §1: $\hat{h}_L \geq C$.

Then $n^2 \hat{h}_L(x) = \hat{h}_L(nx) \geq C \quad \forall n \quad \rightarrow \hat{h}_L(x) \geq 0$.

2) If x is torsion, $nx = 0 \Rightarrow \hat{h}_L(x) = \frac{1}{n^2}\hat{h}_L(nx) = \frac{1}{n^2}\cdot 0 = 0$.

Conversely: if $\hat{h}_L(x) = 0$, consider $\langle x\rangle \subseteq A(K(x))$.

Then $\forall nx \in \langle x\rangle$: $\hat{h}_L(nx) = 0$

Northcott $\Rightarrow \langle x\rangle$ is finite $\Rightarrow$ x is torsion. $\square$

§3 Outlook

Thm. (Ullmo, Zhang; Bogomolov Conjecture)

Let $X \subseteq A$ be a geometrically irreducible subvariety, L symmetric + ample.

Then $\Big(\forall \varepsilon > 0: \{x \in X(\bar{K}) \mid \hat{h}_L(x) \leq \varepsilon\} \subseteq X$ is Zariski dense$\Big) \Leftrightarrow$

$\Leftrightarrow \exists b \in A(\bar{K})_{tors}$, $B \subseteq A_{\bar{K}}$ abelian subvty s.t. $X = b + B$.

Cor. C/K curve (geom. conn + smooth + proj), $A := \mathrm{Jac}\, C$, $g_C \geq 2$.

Then $\exists \varepsilon > 0: \#\{c \in C(\bar{K}) \mid \hat{h}_L(c) \leq \varepsilon\} < \infty$. $\hat{h}_L$ taken from A (note that $C \subseteq A$)

Pf: ↯ Otw: $C = b + E$ for E ell. curve $\subseteq A$. ↯ $\square$

Cor. In ptic, $C(\bar{K}) \cap A(\bar{K})_{tors}$ is finite.

The proof relies on the Equidistribution Thm.

X/K geom irred + proj

Def. $\{x_n\}_n \subseteq X(\bar{K})$ is generic if every subsequence $\{x_{n_i}\}$ is Zariski-dense in X.

Let $(L, \|\cdot\|)$ be adelically metrised (def. later) lb on X, vertically nef (later)

Thm. $\{x_n\} \subseteq X(\bar{K})$ generic, $\sigma: K \hookrightarrow \mathbb{C}$ fixed. Assume

1) $c_1(\bar{L}_\sigma)$ is positive on X_σ, i.e. locally $= i\cdot \sum h_{ij}\, dz_i \times d\bar{z}_j$ where $(h_{ij}) > 0$ pos def herm matrix
2) $h_{(X,\bar{L})}(x) \geq 0 \quad \forall x \in X(\bar{K})$
3) $\lim_{n\to\infty} h_{(X,\bar{L})}(x_n) = 0$

Then

$$\lim_{n\to\infty} \frac{1}{\# O_\sigma(x_n)} \sum_{z \in O_\sigma(x_n)} \delta_z \xrightarrow{\text{weakly}} \frac{c_1(\bar{L}_\sigma)^{\dim X}}{\deg\left(L^{\dim X}\right)}$$

Pf: $V := V(\mathfrak{X})$

Claim. Supp $c_1(\mathcal{L}_i, \|\cdot\|) \subseteq V$ Recall: $c_1(\mathcal{L}_i, \|\cdot\|) = \omega_{(D_2, -\log\|1\|_{\mathcal{L}_i})}$

i.e. $-\log\|1\|_{\mathcal{L}_i}$ is harmonic away from $V \cup D_c$:

$$X^{an} \setminus V = \coprod \mathbb{D}^{an} \amalg \coprod \mathring{A}(a)^{an}$$

Each conn comp C is s.t. $C \subseteq (sp^{an})^{-1}(\mathcal{U})$, $\mathcal{U} \subseteq \mathfrak{X}$ open s.t. $\mathcal{L}_i|_{\mathcal{U}} \simeq \mathcal{O}_{\mathcal{U}}$

In ptic: $\mathcal{O}_{\mathfrak{X}}(D_i)|_C$ trivially metrised formal model $\Rightarrow c_1|_C = 0$.

Claim $g_i|_V = 0$

$D_i / \mathrm{Spf}\, \mathcal{O}_{\mathbb{C}_p}$ is flat, i.e. $1 : \mathcal{O}_{\mathbb{C}_p} \longrightarrow \mathcal{L}_i$ .. $\mathcal{L}_i$ Zariski dense open,

in ptic at each pt of irred comp of $\mathfrak{X}_i$

$\Rightarrow g_1 \cdot c_1(\mathcal{L}_2, \|\cdot\|) = 0 = g_2 \cdot c_1(\mathcal{L}_1, \|\cdot\|)$, i.e. $(g_1 * g_2)(1) = g_1(D_2)$

Now work locally on $\mathfrak{X}$ and by linearity w/ points:

$P, Q \in X(\mathbb{C}_p) \Rightarrow \overline{\{P\}}, \overline{\{Q\}} : \mathrm{Spf}\, \mathcal{O}_{\mathbb{C}_p} \longrightarrow \mathfrak{X}$

$$\begin{cases} \mathcal{O}_{\overline{\{P\}}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{\overline{\{Q\}}} = 0 & \text{if } \overline{P} \neq \overline{Q} \\ \mathcal{O}_{\overline{\{P\}}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{\overline{\{Q\}}} \simeq \mathcal{O}_{\mathbb{C}_p}/(a) & \text{for some } a \in \mathfrak{m}_{\mathbb{C}_p} \Rightarrow \langle \overline{\{P\}}, \overline{\{Q\}} \rangle = v(a) \end{cases}$$

$$-\log\|1\|_{\mathcal{O}_{\mathfrak{X}}(\overline{\{P\}})}(Q) = \begin{cases} \log 1 = 0 & \text{if } \overline{\{P\}} \neq \overline{\{Q\}} \\ v(a) & \text{if } \overline{\{Q\}}^* \mathcal{L} / \mathcal{O}_{\mathbb{C}_p} 1 \simeq \mathcal{O}_{\overline{\{P\}}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{\overline{\{Q\}}} \simeq \mathcal{O}_{\mathbb{C}_p}/(a) \end{cases}$$

Ex. 1) $X := \mathbb{D}^\circ = (\underbrace{\mathrm{Spf}\, \mathcal{O}_{\mathbb{C}_p}[t]}_{\mathfrak{X}})^{ad}_\eta$ $a, b \in X(\mathbb{C}_p)$

$\overline{\{a\}}, \overline{\{b\}}$ def'd by $t-a$, $t-b \in \mathcal{O}_{\mathfrak{X}}(\mathfrak{X})$

$\Rightarrow (a, -\log|t-a|), (b, -\log|t-b|)$, $\mathrm{Spf}\, \mathcal{O}_{\mathbb{C}_p} \longrightarrow \mathfrak{X}$, $\mathcal{O}_{\overline{\{a\}}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{\overline{\{b\}}} \simeq \mathcal{O}_{\mathbb{C}_p}/(a-b)$

$a, b \longleftarrow t$

$\Rightarrow \langle \overline{\{a\}}, \overline{\{b\}} \rangle_{\mathfrak{X}} = \overset{v}{\cancel{v}}(a-b) = -\log|b-a|$

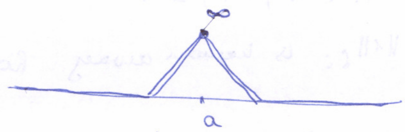
If $a_1, \dots, a_n, b_1, \dots, b_m \in X(\mathbb{C}_p)$, $f = \dfrac{\prod (t-a_i)}{\prod (t-b_j)}$. For $c \in X(\mathbb{C}_p) \setminus \{a_1, \dots, b_1, \dots\}$:

$$-\log|f(c)| = \sum_i \overset{v}{\cancel{v}}(c-a_i) - \sum_j \overset{v}{\cancel{v}}(c-b_j)$$

2) $\mathfrak{X} = \mathbb{P}^1_{\mathcal{O}}$ or more generally: $\mathfrak{X}/\mathrm{Spf}\,\mathcal{O}$ smooth

$\overline{\{a\}} : \mathrm{Spf}\,\mathcal{O} \to \mathfrak{X}$, $\quad \mathcal{L} = \mathcal{O}(a) \xleftarrow{\ s\ } \mathcal{O} \quad b \neq a$

$$(-\log\|s\|)(b) = \begin{cases} 0 & \overline{\{a\}} \neq \overline{\{b\}} \\ v(a-b) & \text{in local coord} \end{cases}$$



$(sp^{an})^{-1}(\overline{\{a\}}) \simeq \mathbb{D}^\circ$

§3 Extension to vertical component

Either work over K discr ... $\Rightarrow$ $\mathfrak{X}$ normal $\Rightarrow$ .. of mult. of irred comp.

or stay over $\mathbb{C}_p$, work with Cartier divisors.

For $\mathcal{I} \subseteq \mathcal{O}_{\mathfrak{X}}$ eff Cartier divisor, $\mathcal{Z} := V(\mathcal{I})$, $\xi \in \mathfrak{X}_k$ generic

Get extn of $|\cdot|$ for $\mathcal{O}_{\mathbb{C}_p}$ to $\mathcal{O}_{\mathfrak{X},\xi}$,

$\mathrm{mult}_\xi \mathcal{Z} := v(a)$ if $\mathcal{O}_{\mathcal{Z},\xi} \simeq \mathcal{O}_{\mathfrak{X},\xi} / a\mathcal{O}_{\mathfrak{X},\xi}$ $\quad a \in \mathcal{O}_{\mathbb{C}_p}$

$\mathcal{I} \subseteq \mathcal{O}_{\mathfrak{X}} \longleftrightarrow 1 : \mathcal{O}_{\mathfrak{X}} \to \mathcal{I}^{-1}$

Prop. $g := -\log\|1\|_{\mathcal{I}^{-1}}$ is the unique function s.t.

1) $g(\xi) = \mathrm{mult}_\xi(\mathcal{Z})$ for $\xi \in V$

2) harmonic on $X^{an} \setminus (V \cup \mathcal{Z}_{\mathbb{C}_p})$

3) g is GF for $\mathcal{Z}_{\mathbb{C}_p}$

Pf: 3) ✓

1) + 2): same as in Prop on flat $\mathcal{Z}/\mathrm{Spf}\,\mathcal{O}$

Cor. $D := \sum_P n_P P \in \mathrm{Div}\,X$, $\mathcal{I}$ eff Cartier div as above, $s : \mathcal{O} \to \mathcal{I}^{-1}$, $|\mathrm{div}(s)_{\mathbb{C}_p}| \cap |\mathrm{Supp}\,D| = \emptyset$. Then $(V(s)_{\mathbb{C}_p}, -\log\|s\|) \cdot (D, -\log\|1\|_{\mathcal{O}_{\mathfrak{X}}(D)}) =$

$$= \sum_P n_P \,\mathrm{len}_{\mathcal{O}_{\mathbb{C}_p}} \mathcal{O}_{\overline{\{P\}}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{\mathcal{Z}}$$

§4 Globalisation

$X/\mathbb{Q}$ sm proj curve

Def. $[D_i, (g_{i,p})_{p\le\infty}] \in \widehat{TCH}^1$ $i = 1,2$, wma $|D_1| \cap |D_2| = \emptyset$. Then let

$$[D_1, (g_{1,p})_p] \cdot [D_2, (g_{2,p})_p] := \sum_{p \le \infty} (g_{1,p} * g_{2,p})(1)$$

Lemma. Well-def'd, sum is finite, descends to $\widehat{TCH}^1(X)$.

$\Rightarrow$ get $\widehat{TCH}^1(X) \otimes \widehat{TCH}^1(X) \to \mathbb{R}$